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Non-orthogonal separable coordinate systems for the flat 4-space Helmholtz equation

E G Kalnins† and Willard Miller Jr‡

† Mathematics Department, University of Waikato, Hamilton, New Zealand
‡ School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

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Abstract. A complete classification of separable non-orthogonal systems for the flat space Helmholtz equation is given. The relation between separability conditions for the various systems and the classification of abelian sub-algebras of the Euclidean symmetry algebra $\epsilon(4)$ is explicitly indicated.

1. Introduction

This paper is a direct application of the results of Boyer *et al* (1978) in which were classified all complex Riemannian spaces for which the Hamilton–Jacobi equation

$$\sum_{k,j=1}^{4} g^{kj}\partial_{x^k}W\partial_{x^j}W = E, \qquad E \neq 0 \tag{1.1}$$

admits a separable solution $W = \Sigma_{k=1}^{4} W^{(k)}(x^k)$. Furthermore, additional conditions to be satisfied by the metric tensor g_{kj} were deduced such that the associated Helmholtz equation

$$g^{-1/2}\sum_{k,j=1}^{4} \partial_{x^k}(g^{1/2}g^{kj}\partial_{x^j}\psi) = \lambda\psi, \qquad \lambda \neq 0 \tag{1.2}$$

be also separable. Here we find all possible non-orthogonal systems in flat space for which the Helmholtz equation admits a separable solution $\psi = \Pi_{j=1}^{4}\psi_j(x^j)$ and relate these systems to Cartesian coordinates.

Thanks to the work of Eisenhart (1934) orthogonal separability for Riemannian spaces is rather well understood. An orthogonal system separates the Hamilton–Jacobi equation if and only if the metric tensor is in Stäckel form with respect to that system. In addition the system separates the Helmholtz equation if and only if the off diagonal elements of the Ricci tensor vanish. (The orthogonal separable systems for the flat space Helmholtz equation are classified in Kalnins and Miller (1978) together with their defining operators in the enveloping algebra of $\epsilon(4)$ and their relation to Cartesian coordinates.)

However, non-orthogonal separable systems are never in Stäckel form and the Ricci condition is no longer necessary and sufficient for Helmholtz separation. In general, relatively few examples of such systems have been found and the theory is not well understood. The detailed results presented here for flat space indicate the complexity of a general solution of the problem for arbitrary Riemannian spaces. Note that our

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results can easily be adapted to determine all separable systems for the Helmholtz equation on any real flat space, e.g. the real Klein–Gordon equation, see Kalnins and Miller (1978). Similarly as we shall show the results also yield all R-separable solutions for the real or complex Schrödinger equation with constant potential

$$\mathrm{i}\partial_t\psi = (-\partial_{xx} - \partial_{yy} + \lambda)\psi. \tag{1.3}$$

The classification of separable systems given in Boyer *et al* (1978) is based on the number of ignorable and essential variables. A variable x^i in a separable system is termed *ignorable* if $\partial_i g_{jk} = 0$ for $1 \leq j, k \leq 4$, i.e. the metric tensor is independent of x^i. Otherwise the variable x^i is *essential*. If the separated ordinary differential equation in the essential variable x^i is first order then x^i is of *type* 1, if second order then x^i is of *type* 2.

To explain our method and to fill a gap in the classification of Boyer *et al* (1978), we treat one example in detail. We consider a separable system for the Hamilton–Jacobi equation (1.1) with two essential variables of type 2 (x^1, x^2), one essential variable of type 1 (x^3), and one ignorable variable (x^4). (This is called a type G equation.) With $W = \Sigma_{j=1}^4 W^{(j)}(x^j)$, $W_j = \partial_j W$, we can write the separated ordinary differential equations in the form

$$\begin{aligned} W_1^2 + f_1 W_4^2 + \lambda_1 a_1 + \lambda_2 b_1 - c_1 E &\equiv \Phi_1 = 0 \\ W_2^2 + f_2 W_4^2 + \lambda_1 a_2 + \lambda_2 b_2 - c_2 E &\equiv \Phi_2 = 0 \\ W_3 W_4 + \lambda_1 a_3 + \lambda_2 b_3 - c_3 E &\equiv \Phi_3 = 0 \\ W_4 \qquad\qquad &= \lambda_3 \end{aligned} \tag{1.4}$$

where f_j, a_j, b_j, c_j are functions of x^j and $\lambda_1, \lambda_2, \lambda_3, E$ are the separation constants. Making the change of variable $x^j = X^j(\tilde{x}^j)$ if necessary, we can assume without loss of generality that $a_1 = b_2 = a_3 = 1$. To relate (1.1) with (1.4) one looks for functions $\Theta_j(x^1, \ldots, x^4)$ such that

$$\sum_{j=1}^{3} \Theta_j \Phi_j \equiv \sum_{i,j=1}^{4} g^{ij} W_i W_j - E, \tag{1.5}$$

identically in the separation constants. (In particular the coefficients of $\lambda_1, \lambda_2, \lambda_3$ should vanish in (1.5) and the coefficient of E should be -1). It is easy to verify that condition (1.5) determines the Θ_j in an essentially unique fashion and leads to the Hamilton–Jacobi equation

$$\begin{aligned} [\mathrm{G3}]Q^{-1}[(a_2b_3 - 1)(W_1^2 + f_1 W_4^2) + (b_1 - b_3)(W_2^2 + f_2 W_4^2) + (1 - a_2 b_1) W_3 W_4] = E, \\ Q = c_1(a_2 b_3 - 1) + c_2(b_1 - b_3) + c_3(1 - a_2 b_1). \end{aligned} \tag{1.6}$$

The most general metric tensor yielding separation of this type can now be read off from (1.6).

The analysis for the Helmholtz equation (1.2) is very similar. Here one looks for separable solutions $\Psi = \Pi_{j=1}^4 \Psi^{(j)}(x^j)$. The separated equations are given by

$$\begin{aligned} \Psi_{11}^{(1)} + h_1 \Psi_1^{(1)} + (f_1 \lambda_3^2 + \lambda_1 a_1 + \lambda_2 b_1 - c_1 \lambda)\Psi^{(1)} &\equiv \Phi_1 \Psi^{(1)} = 0 \\ \Psi_{22}^{(2)} + h_2 \Psi_2^{(2)} + (f_2 \lambda_3^2 + \lambda_1 a_2 + \lambda_2 b_2 - c_2 \lambda)\Psi^{(2)} &\equiv \Phi_2 \Psi^{(2)} = 0 \\ \Psi_3^{(3)} \lambda_3 + (\lambda_1 a_3 + \lambda_2 b_3 - c_3 \lambda)\Psi^{(3)} \equiv \Phi_3 \Psi^{(3)} &\qquad = 0 \\ \Psi_4^{(4)} \qquad\qquad &= \lambda_3 \Psi^{(4)} \end{aligned} \tag{1.7}$$

where $\Psi^{(j)}_{jj} = \partial^2_j \Psi^{(j)}$. To relate (1.2) with (1.7) one looks for functions $\Theta_j(x^1, \ldots, x^4)$ such that

$$\Psi \sum_{j=1}^{3} \Theta_j \Phi_j \equiv g^{-1/2} \sum_{k,j=1}^{4} \partial_k (g^{1/2} g^{kj} \partial_j \Psi) - \lambda \Psi. \tag{1.8}$$

Comparing the coefficients of the second derivative and the constant terms on both sides of (1.8) we find the same solution for the Θ_j and the same metric tensor (g^{jk}) as can be read off from the Hamilton–Jacobi equation (1.6). However, comparison of the coefficients of first derivative terms leads to an additional condition on the metric tensor. (For orthogonal metrics this is called the Robertson condition and amounts to the requirement that the non-diagonal elements of the Ricci tensor vanish, i.e. $R_{ij} = 0$ for $i \neq j$, (Eisenhart 1934). For non-orthogonal metrics, it was shown in Boyer *et al* (1978) that the Ricci tensor condition is not always equivalent to Helmholtz separability.) In this example the additional Helmholtz separability condition is

$$\Gamma \equiv \frac{c_1(a_2b_3 - 1) + c_2(b_1 - b_3) + c_3(1 - a_2b_1)}{(1 - a_2b_1)[(a_2b_3 - 1)(b_1 - b_3)]^{1/2}} = \prod_{i=1}^{3} f_i(x^i), \tag{1.9}$$

i.e. $\partial_{jk} \ln \Gamma = 0$, $1 \leq j < k \leq 3$, and it can be verified that this is equivalent to the requirement $R_{jk} = 0$, $j \neq k$.

In Boyer *et al* (1978) we inadvertently omitted the general system [G3], listing only the special cases [G1]$(b_3 = 0)$ and [G2]$(b_1 = a_2 = 0)$. However, as follows from the preceding paragraph, the statements of all theorems in that paper remain valid.

In particular it is true that any separable system for the flat space Hamilton–Jacobi equation automatically separates the Helmholtz equation. Thus to find the possible non-orthogonal separable flat four-space metrics we need only examine each of the general types of separable metric for a Riemannian space and then require that the curvature tensor corresponding to this metric vanish identically. These curvature equations are frequently very difficult to solve. Fortunately, the conditions can be made tractable by utilisation of the Euclidean symmetry algebra $\epsilon(4)$ of the Helmholtz equation. In particular we show that a knowledge of the abelian sub-algebras of $\epsilon(4)$ with dimensions 1, 2 and 3 enables us to determine the possibilities for ignorable variables corresponding to the metrics and thus to greatly simplify the curvature equations.

The classification of $\epsilon(4)$ abelian sub-algebra is carried out in § 2, and in § 3 all flat space separable metrics are determined. Finally in § 4 we relate each of the separable coordinate systems to Cartesian coordinates. Again our knowledge of $\epsilon(4)$ is helpful in relating each of the ignorable variables to Cartesian coordinates. Furthermore, we show explicitly that the separable solutions for each coordinate system are characterised by a commuting triplet of at most second-order operators in the enveloping algebra of $\epsilon(4)$. The practical and theoretical significance of this operator characterisation for the (special function) solutions is discussed in Miller (1977).

Finally, we mention some of the most interesting features of our results. In Kalnins and Miller (1979) the much easier problem of classifying all non-orthogonal separable systems for the flat 3-space Helmholtz equation was solved and we found that each such system corresponded to an R-separable system for the Schrödinger or heat equation

$$\mathrm{i}\psi_t = -\partial_{xx}\psi + \lambda\psi.$$

Similarly, in four dimensions we find that almost every non-orthogonal separable system for the Helmholtz equation is associated with an R-separable system for (1.3) and that all such systems for (1.3) can be obtained in this way. Indeed our more sophisticated techniques enable us to find many previously unknown R-separable systems for (1.3), particularly those of types [G2] and [E1]–[E3]. However, there is a single non-orthogonal system (4.34) that does not correspond to the heat equation. The significance of this singular system is a topic for future research.

Contrary to a remark in Boyer *et al* (1978) the detailed results in that paper and in the present paper agree with the general theoretical framework for variable separation presented in Woodhouse (1975). However, the definition of variable separation used in Dietz (1976) and Collinson and Fugère (1977) is only a special case of that presented here.

2. Abelian sub-algebras of $\epsilon(4)$

The symmetry algebra of the flat space Helmholtz equation

$$\Delta_4\Psi = \lambda\Psi, \qquad \Delta_4 = \sum_{j=1}^{4} \partial_{z^j z^j} \tag{2.1}$$

is $\epsilon(4)$, the Lie algebra of the complex Euclidean group. A basis for $\epsilon(4)$ is given by

$$\begin{aligned} &P_j = \partial_{z^j}, \qquad j = 1, \ldots, 4 \\ &I_{kl} = z^k\partial_{z^l} - z^l\partial_{z^k}, \qquad I_{lk} = -I_{kl}, \qquad 1 \leqslant k < l \leqslant 4 \end{aligned} \tag{2.2}$$

and the commutation relations are

$$\begin{aligned} &[I_{kl}, I_{mn}] = \delta_{lm}I_{kn} - \delta_{ln}I_{km} + \delta_{kn}I_{lm} - \delta_{km}I_{ln} \\ &[P_j, I_{kl}] = \delta_{jk}P_l - \delta_{jl}P_k, \qquad [P_j, P_k] = 0 \end{aligned} \tag{2.3}$$

where δ_{jk} is the Kronecker delta. For equation (2.1) the symmetry algebra can be identified with the algebra of Killing vectors for the flat space metric. In the general classification of separable coordinate systems for four-dimensional Riemannian manifolds given by Boyer *et al* (1978) the number of ignorable variables in a separable system figured prominently. (The variable x^j in a coordinate system $\{x^1, x^2, x^3, x^4\}$ is *ignorable* if $\partial_{x^j}g_{kl}(x) = 0$, $1 \leqslant k, l \leqslant 4$, i.e. ∂_{x^j} is a symmetry of the associated Helmholtz equation.) It is easy to see that a system with m ignorable variables is associated with an m-dimensional abelian sub-algebra of the symmetry algebra $\mathscr{S}$ of the associated Helmholtz equation. Moreover, since we identify two systems if one can be obtained from the other by an action of the symmetry group, to classify all possibilities for ignorable variables it is necessary and sufficient to determine all equivalence classes of abelian $\mathscr{S}$-sub-algebras under the adjoint action of $\mathscr{S}$.

We now list the classes of abelian sub-algebras for $\epsilon(4)$. The one-dimensional sub-algebras are given in table 1. In each equivalence class we exhibit one represen-

Table 1. One-dimensional sub-algebras of $\epsilon(4)$.

(1)	$K_3 + bJ_3,\ b \in \mathscr{C}$	(6)	$P_3 + iP_4$
(2)	$K_1 + iK_2$	(7)	$K_3 - J_3 + P_1 + iP_2 + b(P_1 - iP_2)$
(3)	$K_1 + iK_2 + J_1 + iJ_2$	(8)	$K_1 + iK_2 + P_1 + iP_2$
(4)	$K_3 + b(J_1 + iJ_2),\ b \neq 0$	(9)	$K_1 + iK_2 + J_1 + iJ_2 + P_1 + iP_2$
(5)	P_1		

tative element. To present the results in their simplest form it is convenient to introduce another basis for the O(4) subalgebra of $\epsilon(4)$. We set

$$
\begin{aligned}
&J_1=\tfrac{1}{2}(I_{23}-I_{14}), && J_2=\tfrac{1}{2}(I_{13}+I_{24}), && J_3=\tfrac{1}{2}(I_{12}-I_{34}),\\
&K_1=\tfrac{1}{2}(I_{23}+I_{14}), && K_2=\tfrac{1}{2}(I_{13}-I_{24}), && K_3=\tfrac{1}{2}(I_{12}+I_{34}),\\
&[J_j, J_k]=\sum_l \epsilon_{jkl}J_l, && [K_j, K_k]=\sum_l \epsilon_{jkl}K_l, &&\\
&[J_j, K_k]=0. && &&
\end{aligned}
\tag{2.4}
$$

Here ϵ_{jkl} is the completely skew-symmetric tensor such that $\epsilon_{123}=+1$.

Suppose u is an ignorable variable belonging to the separable system $\{u, x^2, x^3, x^4\}$. Then we can assume that the symmetry operator $L=\partial_u$ is identical with one of the nine operators listed in table 1. We examine the possibilities and relate u to the standard Cartesian coordinates $\{z^1, z^2, z^3, z^4\}$ in which the metric is

$$
\mathrm{d}s^2=\sum_{j=1}^{4}(\mathrm{d}z^j)^2.
$$

If $L=P_1$ then we can set $z^1=u$, $z^k=z^k(\boldsymbol{x})$, $k=2,3,4$. Thus

$$
\mathrm{d}s^2=\mathrm{d}u^2+\sum_{k=2}^{4}(\mathrm{d}z^k)^2.
\tag{2.5}
$$

If $L=P_3+\mathrm{i}P_4$ then $z^3=u-v$, $z^4=\mathrm{i}(u+v)$ where $v=v(\boldsymbol{x})$, $z^k=z^k(\boldsymbol{x})$, $k=1,2$ and

$$
\mathrm{d}s^2=-4\mathrm{d}u\,\mathrm{d}v+(\mathrm{d}z^1)^2+(\mathrm{d}z^2)^2.
\tag{2.6}
$$

Notice from (2.6) that the system $\{u, \boldsymbol{x}\}$ is necessarily non-orthogonal. No matter what coordinates $\boldsymbol{x}$ are chosen the term $\mathrm{d}u\,\mathrm{d}v$ cannot vanish. Notice also that there is no $(\mathrm{d}u)^2$ term in the metric. These and similar remarks for the remaining seven operators will prove to be of great utility in the classification of separable systems for (2.1).

In table 2 we designate an ignorable coordinate u by 'N' if it is not possible to select this variable so that all cross terms involving $\mathrm{d}u$ vanish from the metric. Thus such variables appear only in non-orthogonal separable systems. All other ignorable coordinates are designated 'O' since they may occur in orthogonal separable systems. In addition we point out the two non-orthogonal coordinates in which there are no $(\mathrm{d}u)^2$ terms in the metric. The variable u is not unique and may be replaced by $u'=u+f(\boldsymbol{x})$ for arbitrary f.

The non-orthogonal 'heat' type variable (6) will prove to be of great interest in the following sections. Note that in terms of the variables (2.6) the effect of the splitting off of u, i.e. assuming a solution of (2.1) in the form $\Psi=\Phi(z^1, z^2, v)\exp(\mathrm{i}\beta u)$ is to reduce (2.1) to the heat (or Schrödinger) equation

$$
\mathrm{i}\beta\partial_v\Phi=(\partial^2_{z^1}+\partial^2_{z^2})\Phi-\lambda\Phi.
\tag{2.7}
$$

A representative basis for each equivalence class of two-dimensional abelian subalgebras of $\epsilon(4)$ is listed in table 3. Again we indicate which associated ignorable variables are intrinsically non-orthogonal.

The corresponding results for three-dimensional abelian sub-algebras are listed in table 4. Sub-algebra (4) does not lead to separable coordinates because the three Lie derivatives are functionally dependent.

Table 2. Metrics and coordinates associated with ignorable variables.

Diagonal operator	Coordinates and metric	Remarks
(1), $b \neq \pm 1$	$z^1 = v_1 \cos \alpha u$, $z^2 = v_1 \sin \alpha u$, $z^3 = v_2 \cos[\beta(u+w)]$, $z^4 = v_2 \sin[\beta(u+w)]$. $\alpha = (1+b)/2$, $\beta = (1-b)/2$, $ds^2 = (v_1^2\alpha^2 + v_2^2\beta^2)\,du^2 + 2v_2^2\beta^2\,du\,dw$ $+ v_2^2\beta^2\,dw^2 + dv_1^2 + dv_2^2$.	N
(1), $b = 1$	$z^1 = v \cos u$, $z^2 = v \sin u$, $ds^2 = dv^2 + v^2\,du^2 + (dz^3)^2 + (dz^4)^2$.	O
(2)	$z^1 = v_1 u$, $z^2 = -iv_1 u + 2v_2$, $z^3 = v_2 u + v_3$, $z^4 = -iv_2 u - iv_3 - 2v_1$.	N, no $(du)^2$ term
(3)	$z^1 = -iv_1 u - \frac{1}{2}i\,e^{v_1}u^2 + iv_2 - i\,e^{v_1}$, $z^2 = -v_1 u - \frac{1}{2}e^{v_1}u^2 + v_3$, $z^3 = v_1 + e^{v_1}u$.	O
(4)	$z^1 + iz^2 = i\,e^{iu/2}(v_1 u + 2v_3)$, $z^1 - iz^2 = -ib^{-1}v_1\,e^{-iu/2}$, $z^1 + iz^4 = -b^{-1}v_1\,e^{iu/2}$, $z^1 - iz^4 = e^{-iu/2}(v_1 u + 2v_2)$.	N
(5)	equation (2.5)	O
(6)	equation (2.6)	N, no $(du)^2$ term
(7)	$z^1 = -(1+b)u + v_1$, $z^2 = i(b-1)u + v_2$, $z^3 = v_3 \cos u$, $z^4 = v_3 \sin u$.	N
(8)	$z^1 = -(1+v_1)u + v_2$, $z^2 = i(v_1 - 1)u - iv_2$, $z^3 = -\frac{1}{2}u^2 + v_3$, $z^4 = -\frac{1}{2}u^2 + 2v_1 - iv_3$.	N
(9)	$z^2 + iz^1 = -2iu$, $z^2 - iz^1 = -2v_1 u + 2v_2 + 2iu^3/3$, $z^3 = v_1 - iu^2$.	N

Table 3. Two-dimensional abelian subalgebras of $\epsilon(4)$.

(1)	J_3, K_3	O
(2)	$J_1 + J_2, K_3$	N
(3)	$J_3 + K_3, P_4$	O
(4)	$J_3 + K_3, P_3 + iP_4$	N
(5)	$K_1 - iK_2, P_3 + iP_4$	N
(6)	$J_1 + iJ_2, K_1 + iK_2$	O
(7)	$K_1 + iK_2, J_1 + iJ_2 + P_3 - iP_4$	N
(8)	$K_1 + iK_2 + J_1 + iJ_2, J_1 + iJ_2 + P_4$	N
(9)	$K_1 + iK_2 + J_1 + iJ_2, P_4$	O
(10)	$-K_1 + iK_2 + J_1 + iJ_2, P_3 + iP_4$	N
(11)	P_1, P_2	O
(12)	$P_1, P_3 + iP_4$	N
(13)	$P_4, J_3 + K_3 + bP_3, b \neq 0$	O
(14)	$P_4, K_1 + iK_2 + J_1 + iJ_2 + P_1 + iP_2$	O
(15)	$P_3 + iP_4, J_3 + K_3 - P_3 + iP_4$	N
(16)	$P_3 + iP_4, -K_1 + iK_2 + J_1 + iJ_2\omega P_1 + iP_2$	N
(17)	$P_3 + iP_4, -K_1 + iK_2 + J_1 + iJ_2 - P_3 + iP_4$	N
(18)	$P_3 + iP_4, -K_1 + iK_2 - P_3 + iP_4$	N
(19)	$P_3 + iP_4, -K_1 + iK_2 - P_1 + iP_2$	N
(20)	$P_3 + iP_4, P_1 - iP_2$	N
(21)	$K_3 + J_3 + bP_3, P_4 + cP_3, bc \neq 0$	N
(22)	$K_1 + iK_2 + P_1 + iP_2, J_1 - iJ_2 + P_1 - iP_2$	N
(23)	$-K_1 + iK_2 + J_1 + iJ_2 - P_3 + iP_4,$ $P_3 + iP_4 + aP_2, a \neq 0$	N

Table 4. Three-dimensional abelian sub-algebras of $\epsilon(4)$.

(1)	P_1, P_2, P_3	O
(2)	P_1, P_2, I_{34}	O
(3)	$P_3+iP_4, K_1-iK_2, J_1+iJ_2$	N
(4)	$P_3+iP_4, P_1+iP_2, K_1-iK_2$	–
(5)	$P_3+iP_4, K_1-iK_2, J_1+iJ_2-P_1-iP_2$	N
(6)	$P_3+iP_4, -I_{14}+iI_{13}, J_1+iJ_2+P_2$	N
(7)	$P_3+iP_4, P_2, -I_{14}+iI_{13}$	N
(8)	P_3+iP_4, P_1, P_2	N
(9)	$P_3+iP_4, P_1, I_{24}-iI_{23}$	N
(10)	$P_3+iP_4, P_1, I_{24}-iI_{23}+P_3-iP_4$	N
(11)	$P_3+iP_4, P_1+iP_2, -K_1+iK_2-P_3+iP_4$	N
(12)	$P_3+iP_4, -K_1+iK_2-P_1+iP_2, J_1+iJ_2-P_1-iP_2$	N
(13)	$P_3+iP_4, P_1-iP_2, J_1+iJ_2+P_1+iP_2+P_3-iP_4$	N
(14)	$P_3+iP_4, P_2, -K_1+iK_2+J_1+iJ_2-P_3+iP_4$	N

Finally, there is only one four-dimensional abelian sub-algebra of $\epsilon(4)$ up to equivalence. One representative has basis P_1, P_2, P_3, P_4 and corresponds to Cartesian coordinates.

3. Classification of non-orthogonal flat space forms for which $\Delta_4\psi = \lambda\psi$ is separable

In this section we classify the inequivalent non-orthogonal differential forms for which (2.1) admits a separation of variables. We showed in Boyer *et al* (1978) that for any four-dimensional Riemannian manifold the possible separable coordinate systems are of eight types depending on the numbers of ignorable variables. Here we determine the number of non-orthogonal separable forms of each type which occur in flat space.

3.1. Forms of type A: *four ignorable variables*

There is only one four-dimensional sub-algebra of $\epsilon(4)$, the Lie algebra of the translation group, and this sub-algebra corresponds to Cartesian coordinates. Hence there is no truly non-orthogonal separable system of this type.

3.2. Forms of type B: *three ignorable variables*

The possible separable systems of this type correspond to the three-dimensional abelian sub-algebras of $\epsilon(4)$ listed in table 4. Sub-algebras (1) and (8) are essentially equivalent to Cartesian coordinates and sub-algebra (2) corresponds to the orthogonal cylindrical coordinates. The remaining ten sub-algebras yield non-orthogonal coordinates, all of heat type. These coordinates can be obtained from taoles 1 and 2. For example, sub-algebra (3) corresponds to the system

$$z^1 = w(u_1-u_2), \qquad z^2 = -iw(u_1+u_2),$$

$$z^3 = -iwu_1u_2+u_3, \qquad z^4 = wu_1u_2+2w+iu_3,$$

and sub-algebra (11) corresponds to

$$z^1+iz^2 = w, \qquad z^1-iz^2 = iu_1^2+2u_2,$$

$$z^3+iz^4 = 2u_1, \qquad z^3-iz^4 = -iu_1w+2u_3.$$

3.3. Forms of type C: two ignorable variables with two essential variables of type 2

Here the Hamilton–Jacobi equation has the form

$$(K_1-K_2)^{-1}[W_1^2+W_2^2+(e_1+e_2)W_3^2+2(h_1+h_2)W_3W_4+(f_1+f_2)W_4^2]=E, \tag{3.1}$$

where $W_i=\partial_{x^i}W$ and $K_i=K_i(x^i)$, $e_i=e_i(x^i)$, etc, and the condition of Helmholtz separability is

$$\partial^2_{x^1x^2}\ln\{(K_1-K_2)^2/[(e_1+e_2)(f_1+f_2)-(h_1+h_2)^2]\}=0. \tag{3.2}$$

Rather than solve this condition directly we observe that for flat space the two Lie symmetries $L_1=\partial_{x^3}$ and $L_2=\partial_{x^4}$ corresponding to the ignorable variables x^3 and x^4 are taken from the list of commuting pairs of symmetries in table 3. For each pair of symmetries from this list there are constraints on the differential form ds^2 and the way in which the differentials dx^3 and dx^4 appear in it. If we run through all the pairs in table 3 and look for differential forms of type C which satisfy the Helmholtz separability condition we obtain only two classes

$$\text{I}\quad ds^2=(K_1-K_2)[(dx^1)^2+(dx^2)^2]+2dx^3\,dx^4-[(f_1-f_2)/(K_1-K_2)](dx^3)^2$$

$$\text{II}\quad ds^2=(K_1-K_2)[(dx^1)^2+(dx^2)^2]+2K_1K_2\,dx^3\,dx^4$$
$$-[(f_1-f_2)/(K_1-K_2)](K_1K_2)^2(dx^3)^2.$$

For metrics of class I with K_1, K_2 not constant we can take

$$ds^2=(x^1-x^2)[X_1(dx^1)^2+X_2(dx^2)^2]+2\,dx^3\,dx^4+[(f_1-f_2)/(x^1-x^2)](dx^3)^2.$$

The flatness condition $R_{1221}=0$ implies (see Eisenhart (1949) for definition of the curvature tensor),

$$X_1^{-1}=a(x^1)^2+bx^1+c, \qquad -X_2^{-1}=a(x^2)^2+bx^2+c.$$

The further condition $R_{1442}=0$ implies $(f_1-f_2)/(x^1-x^2)=x^1+x^2$. The condition $R_{1441}=0$ is satisfied if $a=0$. We then obtain the two differential forms

$$ds^2=(x^1-x^2)[(dx^1)^2/x^1-(dx^2)^2/x^2]+2\,dx^3\,dx^4+(x^1+x^2)(dx^4)^2, \tag{3.3}$$

$$ds^2=(x^1-x^2)[(dx^1)^2-(dx^2)^2]+2\,dx^3\,dx^4+(x^1+x^2)(dx^4)^2. \tag{3.4}$$

Note: If $f_1=f_2=0$ then the metric simplifies to $ds^2=dw^2+2\,dx^3\,dx^4$ where $dw^2(x^1,x^2)$ is a metric in two-dimensional flat space. We shall not trouble to list these well-known metrics.

If $K_2=0$ in metrics of type I we have the differential form

$$ds^2=(dx^1)^2+X_1(dx^2)^2+2\,dx^3\,dx^4+[(f_1-f_2)/X_1](dx^3)^2.$$

We have the flatness condition $R_{1332}=3f_2'X_1'/x_1^2=0$ which requires $X_1=1$. The remaining non-trivial flatness conditions reduce to

$$R_{1331}=\tfrac{1}{2}f_1''=0 \qquad R_{2332}=-\tfrac{1}{2}f_2''=0$$

which gives the metric

$$ds^2=(dx^1)^2+(dx^2)^2+2\,dx^3\,dx^4+(ax^1+bx^2)(dx^4)^2. \tag{3.5}$$

For metrics of type II if K_1, K_2 are not constants then we may write

$$ds^2 = (x^1 - x^2)[X_1(dx^1)^2 + X_2(dx^2)^2] + 2x^1x^2\, dx^3\, dx^4 + (x^1x^2)^2[(f_1 - f_2)/(x^1 - x^2)](dx^3)^2.$$

The flatness condition $R_{1221} = 0$ requires

$$X_1^{-1} = a(x^1)^2 + bx^1 + c, \qquad -X_2^{-1} = a(x^2)^2 + bx^2 + c.$$

The additional condition $R_{3114} = 0$ implies $b = c = 0$. Finally we have the condition $R_{1332} = 0$. This is equivalent to the equation

$$\frac{(x^1)^2 f_1' + (x^2)^2 f_2'}{(x^1x^2)(f_1 - f_2)} = \frac{2}{(x^1 - x^2)}$$

which has the solution $f_1 = 1/(x^1)^2$, $f_2 = 1/(x^2)^2$. We thus have the metric

$$ds^2 = (x^1 - x^2)[(dx^1/x^1)^2 - (dx^2/x^2)] + 2x^1x^2\, dx^3\, dx^4 + (x^1 + x^2)(dx^3)^2. \tag{3.6}$$

If $K_2 = 0$ in differential forms of type II it can be verified that the only possible metrics have three ignorable variables.

3.4. *Forms of type* D: *two ignorable variables with one essential variable of each type*

There are two kinds of systems of this type. For the first of these type [D1], the Hamilton–Jacobi equation is

$$(K_1 - K_2)^{-1}(W_1^2 + 2W_2W_3 + 2b_2W_2W_4 + d_1W_3^2 + 2(f_1 + f_2)W_3W_4 + e_1W_4^2] = E. \tag{3.7}$$

Instead of solving the Helmholtz separability condition

$$\partial_{x^1x^2} \ln\{(K_1 - K_2)^2/[2b_2(f_1 + f_2) - e_1 - b_2^2 d_1]\} = 0,$$

we look for flat space metrics for which the two Lie symmetries $L_1 = \partial_{x^3}$ and $L_2 = \partial_{x^4}$ are taken from the list of commuting pairs of symmetries in table 3. This imposes conditions on the differential form in addition to the Helmholtz separability condition. Proceeding through the list of two dimensional abelian sub-algebras we find that there are no flat space coordinate systems of this type.

For systems of the second kind the Hamilton–Jacobi equation is

$$(K_1 - K_2)^{-1}(W_1^2 + 2W_2W_4 + (d_1 + d_2)W_3^2 + 2f_1W_3W_4 + e_1W_4^2) = E. \tag{3.8}$$

The condition of Helmholtz separability is

$$\partial_{x^1x^2} \ln[(K_1 - K_2)^2/(d_1 + d_2)] = 0. \tag{3.9}$$

This condition is satisfied if $K = K_1 - K_2$, and $d = d_1 + d_2$ are each a function of a single variable only. We consider the various possibilities:

(a) $K = K_2(x^2), \qquad d = d_2(x^2).$

The corresponding differential form is

$$K[(dx^1)^2 + d^{-1}\{(f_1^2 - e_1\, \mathrm{d})(dx^2)^2 + (dx^3)^2 - 2f_1\, dx^2\, dx^3 + 2\, \mathrm{d}\, dx^2\, dx^4\}].$$

The non-trivial flatness conditions $R_{2332}=R_{1221}=R_{1223}=0$ admit the solutions

$$\text{(i)}\quad e_1''=-a,\qquad f_1=0,\qquad K_2=d_2,\qquad a\neq 0,$$

$$\text{(ii)}\quad e_1''=0,\qquad f_1=0,\qquad K_2=d_2 \text{ or } d=1.$$

For the solutions (i) the function K_2 satisfies

$$K_2''=aK_2-\tfrac{3}{2}K_2'^2/K_2=0,$$

which has the solutions

$$K_2=\sec^2(\sqrt{-a}\,x^2),\ \exp(2\sqrt{a}\,x^2).$$

We obtain the differential forms

$$ds^2=(dx^3)^2+[1+(x^2)^2](dx^1)^2+2\,dx^2\,dx^4+[1+(x^2)^2]^{-1}(x^1\,dx^2)^2, \tag{3.10}$$

$$ds^2=(dx^3)^2+x^2(dx^1)^2+2\,dx^2\,dx^4+(x^1\,dx^2)^2/x^2. \tag{3.11}$$

In the case of solutions of type (ii) we find $K_2=1/(x^2)^2$, which gives the differential forms

$$ds^2=(x^2\,dx^1)^2+ax^1(dx^2/x^2)^2+2\,dx^2\,dx^4+(dx^3)^2, \tag{3.12}$$

$$ds^2=(x^2\,dx^1)^2+ax^1(dx^2/x^2)^2+2\,dx^2\,dx^4+(x^2\,dx^3)^2. \tag{3.13}$$

$$\text{(b)}\quad K=K_2(x^2),\qquad d=d_1(x^1).$$

The non-trivial flatness conditions are $R_{1331}=R_{2332}=R_{1221}=R_{1223}=0$ and yield the forms

$$ds^2=[1+(x^2)^2][(dx^1)^2+(x^1\,dx^3)^2]+2\,dx^2\,dx^4+(x^1\,dx^2)^2/[1+(x^2)^2] \tag{3.14}$$

$$ds^2=x^2[(dx^1)^2+(x^1\,dx^3)^2]+2\,dx^2\,dx^4+(x^1\,dx^2)^2/x^2 \tag{3.15}$$

$$ds^2=(x^2)^2[(dx^1)^2+(x^1\,dx^3)^2]+2\,dx^2\,dx^4 \tag{3.16}$$

$$ds^2=(x^2)^2[(dx^1)^2+(dx^3)^2]+2\,dx^2\,dx^4+ax^1(dx^2/x^2)^2. \tag{3.17}$$

All additional forms of this kind correspond to type B.

3.5. Forms of type E*: Two ignorable variables with two essential variables of type 1*

There are three kinds of system to consider. For the first of these, type [E1], the Hamilton–Jacobi equation is

$$(K_1-K_2)^{-1}[2a_1W_1W_3+2W_1W_4+2a_2W_2W_3+2W_2W_4+(c_1-c_2)W_3^2]=E. \tag{3.18}$$

The relevant condition for Helmholtz separability is $\partial_{x^1x^2}\ln[(K_1-K_2)/(a_1-a_2)]=0$. There are two possible solutions to this condition: (a) $K_1-K_2=a_1-a_2$, (b) $K_1-K_2=(a_1a_2)^{-1}(a_1-a_2)$. For systems of kind (a) the non-trivial flatness conditions are $R_{1224}=R_{2114}=R_{1221}=0$. We obtain the two metrics

$$ds^2=2\,dx^3(dx^1-dx^2)+2\,dx^4[(x^1)^2\,dx^2-(x^2)^2\,dx^1], \tag{3.19}$$

$$ds^2=[A(x^1+x^2)+B(x^1+x^2)/(x^1-x^2)+C/(x^1-x^2)]$$
$$\times(dx^1-dx^2)^2+2\,dx^3(dx^1-dx^2)+2\,dx^4(x^1\,dx^2-x^2\,dx^1). \tag{3.20}$$

For systems of kind (b) the flatness conditions are $R_{1223}=R_{2113}=R_{1221}=0$. The corresponding differential form is

$$\mathrm{d}s^2=\left(\frac{A}{x^1}+\frac{B}{(x^1)^2}+\frac{C}{x^2}+\frac{D}{(x^2)^2}\right)(x^1-x^2)^{-1}(x^2\,\mathrm{d}x^1-x^1\,\mathrm{d}x^2)^2$$
$$+2\,\mathrm{d}x^3(x^2\,\mathrm{d}x^1-x^1\,\mathrm{d}x^2)+2\,\mathrm{d}x^4(\mathrm{d}x^2-\mathrm{d}x^1). \qquad (3.21)$$

For systems of type [E2] the Hamilton–Jacobi equation is

$$(K_1-K_2)^{-1}[2W_1W_4+2W_2W_3+2b_2W_2W_4+(c_1-c_2)W_4^2]=E. \qquad (3.22)$$

For Helmholtz separability we must have either (a) $K_2=0$ or (b) $K_1=0$. For systems of kind (a) the flatness conditions are $R_{1223}=R_{2113}=R_{1221}=0$. The corresponding metric has the form

$$\mathrm{d}s^2=\frac{1}{(\sinh\frac{1}{2}x^1)^2}\left[\left(\frac{C}{(x^2)^2}+\frac{D}{x^2}-\frac{A}{\mathrm{e}^{2x^1}}-\frac{B}{\mathrm{e}^{x^1}}\right)(x^2\,\mathrm{d}x^1-\mathrm{d}x^2)^2\right.$$
$$\left.+2\,\mathrm{d}x^3(\mathrm{d}x^2-x^2\,\mathrm{d}x^1)+2\,\mathrm{d}x^1\,\mathrm{d}x^4\right]. \qquad (3.23)$$

There are no new forms of type (b).

For systems of type [E3] the Hamilton–Jacobi equation is

$$(K_1-K_2)^{-1}(2W_1W_4+2W_2W_3+c_1W_3^2+c_2W_4^2)=E. \qquad (3.24)$$

The condition for Helmholtz separability is (without loss of generality) $K_2=0$. The flatness conditions $R_{2113}=R_{1221}=0$ then lead to the possible forms

$$\mathrm{d}s^2=(x^2\,\mathrm{d}x^1/x^1)^2-(\mathrm{d}x^2)^2+2\,\mathrm{d}x^1\,\mathrm{d}x^4+2(x^1)^2\,\mathrm{d}x^2\,\mathrm{d}x^3, \qquad (3.25)$$

$$\mathrm{d}s^2=(x^2\,\mathrm{d}x^1)^2-(x^1\,\mathrm{d}x^2)^2+2\,\mathrm{d}x^1\,\mathrm{d}x^4+2\,\mathrm{d}x^2\,\mathrm{d}x^3, \qquad (3.26)$$

$$\mathrm{d}s^2=x^2(\mathrm{d}x^1/x^1)^2+x_1(\mathrm{d}x^2)^2+2\,\mathrm{d}x^1\,\mathrm{d}x^4+2(x^1)^2\,\mathrm{d}x^2\,\mathrm{d}x^3, \qquad (3.27)$$

$$\mathrm{d}s^2=x^2(\mathrm{d}x^1)^2+x^1(\mathrm{d}x^2)^2+2\,\mathrm{d}x^1\,\mathrm{d}x^4+2\,\mathrm{d}x^2\,\mathrm{d}x^3, \qquad (3.28)$$

$$\mathrm{d}s^2=x^2(\mathrm{d}x^1/x^1)^2+2\,\mathrm{d}x^1\,\mathrm{d}x^4+2(x^1)^2\,\mathrm{d}x^2\,\mathrm{d}x^3. \qquad (3.29)$$

This completes the list of separable metrics of type E.

3.6. *Forms of type* F: *One ignorable variable with three essential variables of type 2*

These forms are all orthogonal.

3.7. *Forms of type* G: *One ignorable variable with one essential variable of type 1 and two of type 2*

There are three cases to consider. For the first of these, type [G1], the Hamilton–Jacobi equation is

$$Q^{-1}[W_1^2+W_2^2+2(l_1-l_2)W_3W_4+(m_1-m_2)W_4^2]=E, \qquad (3.30)$$

where $Q=K_1-K_2+g_3(l_1-l_2)$, $g_3\neq$ constant. The conditions for Helmholtz separability are

$$\partial_{x^ix^j}\ln(Q/l_1-l_2)=0, \qquad 1\leq i<j\leq 3.$$

They are satisfied modulo suitable redefinitions if $K_1 = K_2 = 0$. If l_1 and l_2 are not constants then the flatness conditions $R_{1332} = R_{1221} = R_{1331} = 0$ yield the following differential forms:

$$ds^2 = [1+(x^3)^2]\{c^2(x^1-x^2)[(dx^1)^2/x^1(x^1-1)-(dx^2)^2/x^2(x^2-1)]\} + 2\,dx^3\,dx^4 + (x^1+x^2)(dx^3)^2/[1+(x^3)^2], \tag{3.31}$$

$$ds^2 = [1+(x^3)^2]\{c^2(x^1-x^2)[(dx^1/x^1)^2-(dx^2/x^2)^2]\} + 2\,dx^3\,dx^4 + (x^1+x^2)(dx^3)^2/[1+(x^3)^2], \tag{3.32}$$

$$ds^2 = c^2x^3(x^1-x^2)[(dx^1)^2/x^1(x^1-1)-(dx^2)^2/x^2(x^2-1)] + 2\,dx^3\,dx^4 + (x^1+x^2)(dx^3)^2/x^3, \tag{3.33}$$

$$ds^2 = c^2x^3(x^1-x^2)[(dx^1/x^1)^2-(dx^2/x^2)^2] + 2\,dx^3\,dx^4 + (x^1+x^2)(dx^3)^2/x^3, \tag{3.34}$$

$$ds^2 = c^2(x^3)^2(x^1-x^2)[(dx^1)^2/x^1-(dx^2)^2/x^2] + 2\,dx^3\,dx^4 + (x^1+x^2)(dx^3/x^3)^2, \tag{3.35}$$

$$ds^2 = c^2(x^3)^2(x^1-x^2)[(dx^1)^2-(dx^2)^2] + 2\,dx^3\,dx^4 + (x^1+x^2)(dx^3/x^3)^2, \tag{3.36}$$

$$ds^2 = (x^3)^2(x^1-x^2)[(dx^1)^2/x^1(x^1-1)-(dx^2)^2/x^2(x^2-1)] + 2\,dx^3\,dx^4, \tag{3.37}$$

$$ds^2 = (x^3)^2(x^1-x^2)[(dx^1/x^1)^2-(dx^2/x^2)^2] + 2\,dx^3\,dx^4, \tag{3.38}$$

$$ds^2 = (x^3)^2(x^1-x^2)[(dx^1)^2/x^1-(dx^2)^2/x^2] + 2\,dx^3\,dx^4, \tag{3.39}$$

$$ds^2 = (x^3)^2(x^1-x^2)[(dx^1)^2-(dx^2)^2] + 2\,dx^3\,dx^4. \tag{3.40}$$

To obtain further metrics of this type we choose $l_2 = \text{const}$. The flatness conditions $R_{1221} = R_{1332} = R_{1331} = R_{2332} = 0$ yield the new forms:

$$ds^2 = [1+(x^3)^2][(dx^1)^2+(dx^2)^2] + 2\,dx^3\,dx^4 - [(x^1)^2+(x^2)^2]/[1+(x^3)^2](dx^3)^2, \tag{3.41}$$

$$ds^2 = x^3[(dx^1)^2+(dx^2)^2] + 2\,dx^3\,dx^4 + [(x^1)^2+(x^2)^2](dx^3)^2/4x^3, \tag{3.42}$$

$$ds^2 = (x^3)^2[(dx^1)^2+(dx^2)^2] + 2\,dx^3\,dx^4 + (ax^1+bx^2)(dx^3/x^3)^2. \tag{3.43}$$

For systems of type [G2] the Hamilton–Jacobi equation is

$$Q^{-1}[g_3W_1^2 + l_3W_2^2 + 2W_3W_4 + (u_2l_3+g_3f_1)W_4^2] = E \tag{3.44}$$

where $Q = K_3 + l_3v_2 + g_3r_1$ and the conditions for Helmholtz separability are

$$\partial_{x^ix^j} \ln Q = 0, \qquad 1 \leqslant i < j \leqslant 3.$$

These conditions have the solutions

(a) $r_1 = v_2 = 0$, (b) $K_3 = v_2 = 0$.

In case (a) if we require that g_3 and l_3 are not constants we can apply the flatness conditions $R_{2332} = R_{1331} = 0$ to obtain the metrics

$$ds^2 = (x^3+B)(dx^1)^2 + x^3(dx^2)^2 + 2\,dx^3\,dx^4 + [(x^2)^2/4x^3 + (x^1)^2/4(x^3+B)](dx^3)^2, \qquad B \neq 0, \tag{3.45}$$

$$ds^2 = (x^3+B)(dx^1)^2 + [A^2+(x^3)^2](dx^2)^2 + 2\,dx^3\,dx^4 + [(x^1)^2/4(x^3+B) - (x^2)^2/(A^2+(x^3)^2)](dx_3)^2, \qquad A, B \neq 0, \tag{3.46}$$

$$ds^2 = (x^3+B)(dx^1)^2 + (x^3\,dx^2)^2 + 2\,dx^3\,dx^4 + [Ax^2/(x^3)^2 + x^1/4(x^3+B)](dx^3)^2, \qquad A, B \neq 0, \tag{3.47}$$

$$ds^2 = [(x^3+b)^2+a^2](dx^1)^2 + [1+(x^3)^2](dx^2)^2 + 2\,dx^3\,dx^4 - \{a^2(x^1)^2/[(x^3+b)^2+a^2] + (x^2)^2/[(x^3)^2+1]\}(dx^3)^2, \qquad b \neq 0,\ a \neq 1, \tag{3.48}$$

$$ds^2 = (x^3+b)^2(dx^1)^2 + [1+(x^3)^2](dx^2)^2 + 2\,dx^3\,dx^4 + \{-(x^2)^2/[1+(x^3)^2] + Ax^1/(x^3+b)^2\}(dx^3)^2, \qquad A, b \neq 0, \tag{3.49}$$

$$ds^2 = (x^3+b)^2(dx^1)^2 + (x^3\,dx^2)^2 + 2\,dx^3\,dx^4 + [Ax^1/(x^3+b)^2 + Bx^2/(x^3)^2](dx^3)^2. \tag{3.50}$$

In case (b) the flatness conditions $R_{2332} = R_{1331} = 0$ lead to the metrics

$$ds^2 = (dx^1)^2 + x^3(dx^2)^2 + [Ax^1 + (x^2)^2/4x^3](dx^3)^2 + 2\,dx^3\,dx^4, \qquad A \neq 0, \tag{3.51}$$

$$ds^2 = (dx^1)^2 + (1+(x^3)^2)(dx^2)^2 + \{Ax^1 - (x^2)^2/[1+(x^3)^2]\}(dx^3)^2 + 2\,dx^3\,dx^4, \qquad A \neq 0, \tag{3.52}$$

$$ds^2 = (dx^1)^2 + (x^3\,dx^2)^2 + [Ax^1 + Bx^2/(x^3)^2](dx^3)^2 + 2\,dx^3\,dx^4. \tag{3.53}$$

The possibility l_3 = constant from case (a) is included here. If both l_3 and g_3 are constant there are two ignorable variables.

It can be checked that the general type [G3] system leads to no new separable coordinates in flat space.

3.8. Forms of type H: *No ignorable variables*

Systems of this type are necessarily orthogonal.

4. Non-orthogonal coordinate systems for which $\Delta_4\psi = \lambda\psi$ is separable

Here we examine the coordinate systems, separation equations and operator characterisation of the differential forms classified in the previous section. We give these results in summarised form, working out some particular cases in detail for purposes of illustration. A complete and detailed discussion of all systems would lengthen this section unnecessarily. We group the various coordinate systems into classes which are closely related. In all but the last case the systems we find correspond to the embedding of the heat equation into the complex Helmholtz equation.

4.1. For the differential form (3.41), type [G1], the relation between the separable and Cartesian coordinates is

$$\begin{aligned} z^1 &= x^1[1+(x^3)^2]^{1/2}, \qquad z^2 = x^2[1+(x^3)^2]^{1/2}, \\ z^3 - iz^4 &= x^4 - \tfrac{1}{2}x^3[(x^1)^2+(x^2)^2], \qquad z^3 + iz^4 = 2x^3. \end{aligned} \tag{4.1}$$

The complex Helmholtz equation reads

$$\Delta_4\psi = \{(1+(x^3)^2)^{-1}[(\partial_{x^1})^2+(\partial_{x^2})^2]+(x^1\partial_{x^4})^2+(x^2\partial_{x^4})^2+2\partial_{x^3x^4}\}\psi=\lambda\psi,$$

so the separation equations are

$$E_4' = l_3E_4, \qquad (4.2)$$

$$\{\partial_{x^1x^1}+\partial_{x^2x^2}+[(x^1)^2+(x^2)^2]l_3^2\}E_1E_2 = l_1E_1E_2, \qquad (4.3)$$

$$[\partial_{x^2x^2}+(l_3x^2)^2]E_2 = l_2E_2, \qquad (4.4)$$

$$2l_3[1+(x^3)^2]E_3' = \{\lambda[1+(x^3)^2]-l_1-l_2\}E_3, \qquad (4.5)$$

where we have sought a solution of the form $\psi=\Pi_{k=1}^4 E_k(x^k)$. The operators $L_j (j=1,2,3)$ whose eigenvalues are the separation constants l_j, are

$$L_1 = P_1^2+P_2^2+\tfrac14(I_{31}+iI_{41})^2+\tfrac14(I_{32}+iI_{42})^2,$$
$$L_2 = P_2^2+\tfrac14(I_{32}+iI_{42})^2, \qquad L_3=\tfrac12(P_3+iP_4). \qquad (4.6)$$

This coordinate system corresponds to an embedding of the heat equation (2.7) into the four-dimensional Helmholtz equation. In the coordinates given above we have clearly redefined the variable u in terms of the new ignorable variable x^4. (Note that any transformation $x^4\to x^4+f(x^1,x^3,x^3)$ still leaves x^4 ignorable.) This however means that if we restrict ourselves to (2.7) then we normally must consider R-separable coordinate systems. Embedded in four dimensional complex space, (2.7) corresponds to choosing eigenfunctions of P_3+iP_4 with eigenvalue $i\beta$. The remaining separation constants for all heat equation coordinate systems are second order symmetric operators in the enveloping algebra of the Galilei subgroup of $\epsilon(4)$ with generators $P_1, P_2, I_{31}+iI_{41}, I_{32}+iI_{42}, I_{12}$. Similar separable coordinates are obtained by substituting for x^1 and x^2 all inequivalent complex coordinates for which the harmonic oscillator equation (4.3) admits a separation of variables. We give the change of variables together with the operator L_2 specifying these various systems. For all of them

$$L_1 = P_1^2+P_2^2+\tfrac14(I_{31}+iI_{41})^2+\tfrac14(I_{32}+iI_{42})^2.$$

(2) $x^1\to x^1\cos x^2, \qquad x^2\to x^1\sin x^2, \qquad L_2=I_{12}^2.$ (3.14)

(3) $x^1\to(x^1x^2)^{1/2}, \qquad x^2\to[(x^1-1)(1-x^2)]^{1/2},$

$$L_2=P_1^2+I_{12}^2+\tfrac14(I_{31}+iI_{41})^2. \qquad (3.31)$$

(4) $x^1+ix^2\to(x^1x^2)^{1/2}, \qquad x^1-ix^2\to(x^1/x^2)^{1/2}+(x^2/x^1)^{1/2},$

$$L_2=-I_{12}^2+(P_1+iP_2)^2+\tfrac14[I_{31}+iI_{41}+i(I_{32}+iI_{42})]^2. \qquad (3.32)$$

4.2. A related class of coordinate systems follows from the differential form (3.42). A suitable choice of coordinates is

$$z^1=x^1(x^3)^{1/2}, \qquad z^2=x^2(x^3)^{1/2}$$
$$z^3-iz^4=x^4-\tfrac14[(x^1)^2+(x^2)^2], \qquad z^3+iz^4=2x^3. \qquad (4.7)$$

The defining operators L_i are

$$L_1=\tfrac14\{P_1, I_{31}+iI_{41}\}+\tfrac14\{P_2, I_{32}+iI_{42}\}, \qquad L_2=\tfrac14\{P_2, I_{32}+iI_{42}\},$$
$$L_3=\tfrac12(P_3+iP_4). \qquad (4.8)$$

(Here, $\{A, B\} = AB + BA$.) Again this system corresponds to an embedding of (2.7) into the Helmholtz equation (1.1). Similar coordinates are obtained in the same way as for systems of type 1. For all these possibilities

$$L_1 = \tfrac{1}{4}\{P_1, I_{31} + \mathrm{i}I_{41}\} + \tfrac{1}{4}\{P_2, I_{32} + \mathrm{i}I_{42}\}.$$

$$(2)\quad x^1 \to x^1 \cos x^2, \qquad x^2 \to x^1 \sin x^2, \qquad L_2 = I_{12}^2. \tag{3.15}$$

$$(3)\quad x^1 \to (x^1x^2)^{1/2}, \qquad x^2 \to [(x^1-1)(1-x^2)]^{1/2},$$

$$L_2 = I_{12}^2 + \tfrac{1}{4}\{P_1, I_{31} + \mathrm{i}I_{41}\}. \tag{3.33}$$

$$\begin{gathered}(4)\quad x^1 + \mathrm{i}x^2 \to (x^1x^2)^{1/2}, \qquad x^1 - \mathrm{i}x^2 \to (x^1/x^2)^{1/2} + (x^2/x^1)^{1/2}, \\ L_2 = -I_{12}^2 - \tfrac{1}{4}\{P_1 + \mathrm{i}P_2, I_{31} + \mathrm{i}I_{41} + \mathrm{i}(I_{32} + \mathrm{i}I_{42})\}.\end{gathered} \tag{3.34}$$

4.3. For this class the prototype differential form is (3.43). A suitable choice of coordinates is

$$\begin{gathered} z^1 = x^1x^3 - b/4x^3, \qquad z^2 = x^2x^3 - a/4x^3, \\ z^3 - \mathrm{i}z^4 = x^4 - \tfrac{1}{2}x^3[(x^1)^2 + (x^2)^2] - (ax^2 + bx^1)/4x^3 + (a^2+b^2)/96(x^3)^3, \\ z^3 + \mathrm{i}z^4 = 2x^3. \end{gathered} \tag{4.9}$$

The operators L_i are

$$\begin{gathered} L_1 = \tfrac{1}{4}(I_{31} + \mathrm{i}I_{41})^2 - \tfrac{1}{2}bP_1(P_3 + \mathrm{i}P_4) + \tfrac{1}{4}(I_{32} + \mathrm{i}I_{42})^2 - \tfrac{1}{2}aP_2(P_3 + \mathrm{i}P_4), \\ L_2 = \tfrac{1}{4}(I_{32} + \mathrm{i}I_{42})^2 - \tfrac{1}{2}aP_2(P_3 + \mathrm{i}P_4), \qquad L_3 = \tfrac{1}{2}(P_3 + \mathrm{i}P_4). \end{gathered} \tag{4.10}$$

This coordinate system again corresponds to an embedding of (2.7) into the Helmholtz equation. Similar coordinate systems are obtained by substituting for x^1 and x^2 all inequivalent complex coordinates for which the linear potential Schrödinger equation

$$[\partial_{x^1x^1} + \partial_{x^2x^2} - (ax^2 + bx^1)l_3^2]\Phi = l_1\Phi, \qquad \Phi = E_1E_2 \tag{4.11}$$

admits a separation of variables. In considering these possibilities it may be that the constants a, b must have certain values, e.g. $a = 0$. We now specify the remaining systems by giving the change of variables together with the operator L_2 and restrictions on a, b, if any. For all these operators

$$L_1 = \tfrac{1}{4}(I_{31} + \mathrm{i}I_{41})^2 + \tfrac{1}{4}(I_{32} + \mathrm{i}I_{42})^2 - \tfrac{1}{2}bP_1(P_3 + \mathrm{i}P_4) - \tfrac{1}{2}aP_2(P_3 + \mathrm{i}P_4).$$

$$\begin{gathered}(2)\quad x^1 \to \tfrac{1}{2}[(x^1)^2 - (x^2)^2], \qquad x^2 \to x^1x^2; \qquad a = 0, \\ L_2 = \tfrac{1}{2}\{I_{12}, I_{32} + \mathrm{i}I_{42}\} + \tfrac{1}{2}bP_2^2.\end{gathered} \tag{3.35}$$

$$\begin{gathered}(3)\quad x^1 + \mathrm{i}x^2 \to \tfrac{1}{2}(x^1 - x^2)^2, \qquad x^1 - \mathrm{i}x^2 \to (x^1 + x^2), \qquad a = -\mathrm{i}b, \\ L_2 = -\tfrac{1}{4}\{I_{12}, I_{32} + \mathrm{i}I_{31} + \mathrm{i}(I_{42} + \mathrm{i}I_{41})\} - \tfrac{1}{16}[I_{31} + I_{24} + \mathrm{i}(I_{41} + I_{32})]^2 \\ + \tfrac{1}{4}b[(P_1 - \mathrm{i}P_2)^2 + (P_1 + \mathrm{i}P_2)(P_3 + \mathrm{i}P_4)].\end{gathered} \tag{3.36}$$

All remaining systems correspond to $a = b = 0$:

$$(4)\quad x^1 \to x^1 \cos x^2, \qquad x^2 \to x^1 \sin x^2, \qquad L_2 = I_{12}^2. \tag{3.16}$$

$$(5)\quad x^1 \to (x^1x^2)^{1/2}, \qquad x^2 \to [(x^1-1)(1-x^2)]^{1/2},$$

$$L_2 = I_{12}^2 + \tfrac{1}{4}(I_{31} + \mathrm{i}I_{41})^2, \tag{3.37}$$

$$(6)\quad x^1+\mathrm{i}x^2\to(x^1x^2)^{1/2},\qquad x^1-\mathrm{i}x^2\to(x^1/x^2)^{1/2}+(x^2/x^1)^{1/2}$$
$$L_2=I_{12}^2+\tfrac{1}{4}[I_{31}+I_{24}+\mathrm{i}(I_{41}+I_{32})]^2. \tag{3.38}$$

4.4. For the next class the prototype differential form is (3.5). A suitable choice of coordinates is

$$z^1=x^1-\tfrac{1}{4}b(x^3)^2,\qquad z^2=x^2-\tfrac{1}{4}a(x^3)^2,$$
$$z^3-\mathrm{i}z^4=x^4-\tfrac{1}{2}(ax^2+bx^1)x^3+\tfrac{1}{24}(a^2+b^2)(x^3)^3,\qquad z^3+\mathrm{i}z^4=2x^3. \tag{4.12}$$

The operators L_i are

$$L_2=P_2^2+\tfrac{1}{8}a\{P_3+\mathrm{i}P_4,I_{23}+\mathrm{i}I_{24}\}.\qquad L_3=\tfrac{1}{2}(P_3+\mathrm{i}P_4), \tag{4.13}$$

with L_1 given below. This system again corresponds to an embedding of the heat equation into (2.1). Similar coordinates are obtained in the same way as for systems of type 3. For all these systems

$$L_1=P_1^2+P_2^2+\tfrac{1}{8}b\{P_3+\mathrm{i}P_4,I_{13}+\mathrm{i}I_{14}\}+\tfrac{1}{8}a\{P_3+\mathrm{i}P_4,I_{23}+\mathrm{i}I_{24}\}, \tag{4.14}$$

$$(2)\quad x^1\to\tfrac{1}{2}[(x^1)^2-(x^2)^2],\qquad x^2\to x^1x^2,\qquad a=0,$$
$$L_2=-\{I_{12},P_2\}+\tfrac{1}{8}b(I_{32}+\mathrm{i}I_{42})^2. \tag{3.3}$$

$$(3)\quad x^1+\mathrm{i}x^2\to\tfrac{1}{2}(x^1-x^2)^2,\qquad x^1-\mathrm{i}x^2\to x^1+x^2,\qquad a=-\mathrm{i}b,$$
$$L_2=\tfrac{1}{2}\{I_{12},P_1-\mathrm{i}P_2\}-\tfrac{1}{4}(P_1+\mathrm{i}P_2)^2+\tfrac{1}{16}b[I_{31}+\mathrm{i}I_{41}-\mathrm{i}(I_{32}+\mathrm{i}I_{42})]^2$$
$$+\{P_3+\mathrm{i}P_4,I_{13}+\mathrm{i}I_{23}+\mathrm{i}(I_{14}+\mathrm{i}I_{24})\}. \tag{3.4}$$

The remaining systems correspond to $a=b=0$.

$$(4)\quad (\text{type B})\quad x^1\to x^1\cos x^2,\qquad x^2\to x^1\sin x^2,\qquad L_2=I_{12}^2.$$

$$(5)\quad x^1\to(x^1x^2)^{1/2},\qquad x^2\to[(x^1-1)(1-x^2)]^{1/2},\qquad L_2=I_{12}^2+P_2^2.$$

$$(6)\quad x^1+\mathrm{i}x^2\to(x^1x^2)^{1/2},\qquad x^1-\mathrm{i}x^2\to(x^1/x^2)^{1/2}+(x^2/x^1)^{1/2},$$
$$L_2=I_{12}^2+(P_1+\mathrm{i}P_2)^2.$$

For these last two systems see the remark following equation (3.4).

4.5. These systems correspond to type [G2] forms. The coordinates are

$$z^1=F_k(x^1,x^3),\qquad z^2=F_j(x^2,x^3),$$
$$z^3-\mathrm{i}z^4=x^4+H_k(x^1,x^3)+H_j(x^2,x^3),\qquad z^3+\mathrm{i}z^4=2x^3, \tag{4.15}$$

where the functions F_k, H_k are one of the four possibilities

$$(\text{i})\quad F_1(u,x^3)=u-\tfrac{1}{4}a_1(x^3+b_1)^2,$$
$$H_1(u,x^3)=\tfrac{1}{2}a_1u(x^3+b_1)-\tfrac{1}{24}a_1^2(x^3+b_1)^3.$$

$$(\text{ii})\quad F_2(u,x^3)=u(x^3+b_2)-\tfrac{1}{4}a_2/(x^3+b_2),$$
$$H_2(u,x^3)=-\tfrac{1}{2}(x_3+b_2)u^2-\tfrac{1}{4}a_2u/(x^3+b_2)+\tfrac{1}{96}a_2^2/(x^3+b_2)^2.$$

$$(\text{iii})\quad F_3(u,x^3)=u[(x^3+b_3)^2+a_3^2]^{1/2},\qquad H_3(u,x^3)=-\tfrac{1}{2}(x^2+b_3)u^2.$$

$$(\text{iv})\quad F_4(u,x^3)=u(x^3+b_4)^{1/2},\qquad H_4(u,x^3)=0.$$

Here the numbers a_j $(j=1,2,3)$ and b_k $(k=1,2,3,4)$ are arbitrary. In certain instances when $j=k$ in the coordinates and the constants are equal then we recover systems already included in types 1–4. Coordinate systems of this type correspond to the embedding of two different separable systems for the equation $\theta_{xx}+\mathrm{i}\theta_t=\beta\theta$ into (2.7). These systems were not found in all generality in our previous analysis of (2.7), (Boyer *et al* 1975) as our earlier notation of separability was inadequate. We illustrate with a non-trivial example. Consider the metric (3.46). The coordinates are

$$z^1=x^1(x^3+B)^{1/2},\qquad z_2=x^2[A^2+(x^3)^2]^{1/2},$$
$$z^3-\mathrm{i}z^4=x^4-\tfrac{1}{4}(x^1)^2-\tfrac{1}{2}x^3(x^2)^2,\qquad z^3+\mathrm{i}z^4=2x^3, \tag{4.16}$$

and the Helmholtz equation assumes the form

$$\Delta_4\psi=\{(x^3+B)^{-1}[\partial_{x^1x^1}-\tfrac{1}{2}x^1\partial_{x^4})^2]+[A^2+(x^3)^2]^{-1}[\partial_{x^2x^2}+(a^2x^2\partial_{x^4})^2]+2\partial_{x^3x^4}$$
$$+[[2(x^3+B)]^{-1}+x^3/[A^2+(x^3)^2]]\partial_{x^4}\}\psi=\lambda\psi. \tag{4.17}$$

The defining operators are

$$L_1=\tfrac{1}{2}(P_3+\mathrm{i}P_4),\qquad L_2=\tfrac{1}{4}\{I_{31}+\mathrm{i}I_{41},P_1\}+BP_1^2,\qquad L_3=P_2^2+\tfrac{1}{4}A^2(I_{32}+\mathrm{i}I_{42})^2. \tag{4.18}$$

4.6. These systems correspond to differential forms of type E. We give the details for one case and list some of the coordinates for other cases. Consider the differential form (3.28). A suitable choice of coordinates is

$$z^1+\mathrm{i}z^2=x^1x^2-\tfrac{1}{2}(x^2)^2+2x^4,\qquad z^1-\mathrm{i}z^2=x^1,$$
$$z^3+\mathrm{i}z^4=x^1x^2-\tfrac{1}{2}(x^1)^2+2x^3,\qquad z^3-\mathrm{i}z^4=x^2. \tag{4.19}$$

The complex Helmholtz equation has the form

$$\Delta_4\psi=(2\partial_{x^1x^4}+2\partial_{x^2x^3}+x^1\partial_{x^3x^3}+x^2\partial_{x^4x^4})\psi=\lambda\psi. \tag{4.20}$$

The defining operators are

$$L_1=P_1-\mathrm{i}P_2,\qquad L_2=P_3-\mathrm{i}P_4,$$
$$L_3=P_1^2+P_2^2+\tfrac{1}{2}\{I_{24}+\mathrm{i}I_{23}+\mathrm{i}(I_{14}+\mathrm{i}I_{13}),P_1-\mathrm{i}P_2+P_3-\mathrm{i}P_4\}. \tag{4.21}$$

(2) For metric (3.29) the coordinates read

$$z^1-\mathrm{i}z^2=2x^1x^2,\qquad z^1+\mathrm{i}z^2=x^1x^3-(4x^1)^{-1},$$
$$z^3+\mathrm{i}z^4=x^1,\qquad z^3-\mathrm{i}z^4=-\tfrac{1}{2}x^2/x^1-2x^2x^3x^1+2x^4, \tag{4.22}$$

and the defining operators are

$$L_1=P_3-\mathrm{i}P_4,\qquad L_2=\tfrac{1}{2}[I_{24}+\mathrm{i}I_{14}+\mathrm{i}(I_{23}+\mathrm{i}I_{13})]$$
$$L_3=-(I_{14}+\mathrm{i}I_{13})^2-(I_{24}+\mathrm{i}I_{23})^2-\tfrac{1}{2}(P_1-\mathrm{i}P_2)(P_3-\mathrm{i}P_4). \tag{4.23}$$

(3) For the metric (2.25) a suitable choice of coordinates is

$$z^1-\mathrm{i}z^2=2x^1x^2,\qquad z^1+\mathrm{i}z^2=x^1x^3-\tfrac{1}{2}x^2/x^1,$$
$$z^3-\mathrm{i}z^4=x^1,\qquad z^3+\mathrm{i}z^4=-2x^1x^2x^3+2x^4. \tag{4.24}$$

The operators are

$$L_1 = P_3 - iP_4, \qquad L_2 = \tfrac{1}{2}(I_{24} + iI_{14} + i(I_{23} + iI_{13})),$$
$$L_3 = -(I_{41} + iI_{31})^2 - (I_{42} + iI_{32})^2 - \tfrac{1}{4}(P_1 - iP_2)^2. \tag{4.25}$$

(4) For the metric (3.26) a suitable choice of coordinates is

$$z^1 + iz^2 = x^1(x^2)^2 + 2x^4, \qquad z^1 - iz^2 = x^1,$$
$$z^3 + iz^4 = -x^2(x^1)^2 + 2x^3, \qquad z^3 - iz^4 = x^2. \tag{4.26}$$

The operators are

$$L_1 = P_1 - iP_2, \qquad L_2 = P_3 - iP_4,$$
$$L_3 = P_1^2 + P_2^2 + [I_{31} + I_{24} + i(I_{41} + I_{23})]^2. \tag{4.27}$$

(5) For the metric (3.27) the coordinates are

$$z^1 - iz^2 = 2x^2x^1, \qquad z^3 - iz^4 = x^1,$$
$$z^1 + iz^2 = x^1x^3 - (4x^1)^{-1} + \tfrac{1}{2}x^2, \tag{4.28}$$
$$z^3 + iz^4 = -\tfrac{1}{2}x^2/x^1 - 2x^3x^2x^1 - \tfrac{1}{2}(x^2)^2 + 2x^4,$$

and the operators are

$$L_1 = P_3 - iP_4, \qquad L_2 = I_{24} + iI_{23} + i(I_{14} + iI_{13}),$$
$$L_3 = -(I_{41} + iI_{31})^2 - (I_{42} + iI_{32})^2 + \tfrac{1}{8}\{P_1 - iP_2, I_{31} + I_{24} + i(I_{14} + I_{23})\} \tag{4.29}$$
$$-\tfrac{1}{2}(P_1 - iP_2)(P_3 - iP_4).$$

(6) For the metric (3.19) coordinates are

$$z^1 - iz^2 = -2x^1x^2, \qquad z^1 + iz^2 = (x^1 - x^2)x^4,$$
$$z^3 - iz^4 = x^1 - x^2, \qquad z^3 + iz^4 = 2x^1x^2x^4 + 2x^3, \tag{4.30}$$

and the operators are

$$L_1 = \tfrac{1}{2}[I_{24} + iI_{23} + i(I_{14} + iI_{13})], \qquad L_2 = P_3 - iP_4,$$
$$L_3 = (I_{24} + iI_{23})^2 + (I_{14} + iI_{13})^2 + \tfrac{1}{2}i\{I_{12}, P_1 - iP_2\}. \tag{4.31}$$

The remaining systems are more complicated. We give here a typical example. For the metric (3.20) a suitable choice of coordinates is

$$z^1 - iz^2 = x^2 - x^1, \qquad z^3 - iz^4 = x^1 + x^2,$$
$$z^1 + iz^2 = (x^1 + x^2)x^4 - 2x^3 + G(x^1, x^2), \tag{4.32}$$
$$z^3 + iz^4 = (x^1 - x^2)x^4 + H(x^1, x^2).$$

Where the functions G and H have the form

$$G = A[(x^2)^2 - (x^1)^2] - \tfrac{1}{2}B(x^1 + x^2)[1 + 2\ln(x^1 - x^2)] - C\ln(x^1 - x^2)$$
$$H = -\tfrac{1}{2}A(x^1 - x^2)^2 + \tfrac{1}{2}B(x^1 - x^2)[1 - 2\ln(x^1 - x^2)].$$

The separation equations and operators describing separation can be computed but the results are rather long formulae. We note, however, that all such coordinate systems generate new R-separable solutions of (2.7) as they correspond to systems related to the embedding of this equation into the Helmholtz equation.

4.7. Finally we list the single non-orthogonal system that does not correspond to an embedding of (2.7) into the Helmholtz equation. The differential form is (3.6),

$$ds^2 = \tfrac{1}{4}(x^1 - x^2)[(dx^1/x^1)^2 - (dx^2/x^2)^2] + 2x^1x^2\, dx^3\, dx^4 - (x^1 + x^2)(dx^4)^2. \tag{4.33}$$

A suitable choice of coordinates is

$$\begin{aligned} z^1 + iz^2 &= (x^1x^2)^{1/2} \cosh x^4, \qquad z^3 + iz^4 = (x^1x^2)^{1/2} \sinh x^4, \\ z^1 - iz^2 &= [(x^1/x^2)^{1/2} + (x^2/x^1)^{1/2}] \cosh x^4 - x^3(x^1x^2)^{1/2} \sinh x^4, \\ z^3 - iz^4 &= -[(x^1/x^2)^{1/2} + (x^2/x^1)^{1/2}] \sinh x^4 + x^3(x^1x^2)^{1/2} \cosh x^4. \end{aligned} \tag{4.34}$$

The Helmholtz equation becomes

$$\begin{aligned} \Delta_4\psi = \{&4(x^1 - x^2)^{-1}[(x^1)^{-1}\partial_{x^1}((x^1)^3\partial_{x^1}) - x^{-2}\partial_{x^2}((x^2)^3\partial_{x^2})] \\ &+ [(x^1 + x^2)/(x^1x^2)^2]\partial_{x^3x^3} + 2(x^1x^2)^{-1}\partial_{x^3x^4}\}\psi = \lambda\psi, \end{aligned} \tag{4.35}$$

and the defining operators are

$$\begin{aligned} L_1 &= I_{13} + iI_{23} + i(I_{14} + iI_{24}), \qquad L_2 = i(I_{41} + I_{23}), \\ L_3 &= I_{12}^2 + I_{34}^2 + (P_1 + iP_2)^2 + (P_3 + iP_4)^2 - \{I_{12}, I_{34}\} - (I_{31} + I_{42})^2 - (I_{23} + I_{41})^2. \end{aligned} \tag{4.36}$$

This completes our treatment of the non-orthogonal systems for which $\Delta_4\psi = \lambda\psi$ admits a separation of variables. In all but one case the coordinate systems correspond to R-separable systems for the Schrödinger equation (2.7) embedded in the Helmholtz equation. We have systematically found all R-separable systems for (2.7) by embedding it in (2.1).

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